ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH TURNING POINTS

REVIEW OF THE LITERATURE

Technical Report 1

Prepared under contract Nonr-220(11)

for the
Office of Naval Research

Reference no. NR 043-121

Department of Mathematics
California Institute of Technology
Pasadena
1953

ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH TURNING POINTS

REVIEW OF THE LITERATURE

Technical Report 1

Prepared under contract Nonr-220(11) for the Office of Naval Research

H. F. Bohnenblust, *Director*

T. M. Apostol, A. Erdélyi, *Senior Investigators*

L. D. Berkovitz, *Research Fellow*

M. Kennedy, J. L. McGregor, *Research Assistants*

Reference no. NR 043-121

Department of Mathematics
California Institute of Technology
Pasadena
1953

Asymptotic solutions of ordinary linear differential equations show peculiarities in the neighborhood of certain exceptional points, called transition or turning points. In the case of a real independent variable, the typical behavior is a change from oscillatory to monotonic solutions as the independent variable passes through a turning point. The classical theory of asymptotic solutions breaks down in the neighborhood of such a point, and special investigations are required.

In the present report, papers dealing with this situation are listed alphabetically, together with a brief description of their contribution to the subject of this survey. Basically, only mathematical papers have been listed, but a few exceptions occur. The early papers on the W.K.B. method have been included for historical reasons even though they do not contain a mathematical theory. Certain papers on physical problems were included because in some way they contributed to the mathematical theory. Papers on special functions were included if their results were derived from the differential equation. There is a large number of papers in which the behavior of special functions in the neighborhood of a transition point is investigated by means of the method of steepest descent applied to their integral representations, or by means of some other method which is not based on the differential equation. Such papers have been excluded from the present survey.

Most of the papers listed here deal with differential equations of the second order. Very little work has been done on differential equations of higher order, or on systems of differential equations of the first order with turning points; a few papers whose methods or results seem to offer a good starting point for further research in these directions have been included.

Efforts will be made to keep this bibliography up to date. Corrections and additions will be welcome.

Brillouin, L. Remarques sur la Mécanique Ondulatoire. J. Phys. Radium 7, 353-368 (1926).

The author obtains a formal solution of the 3-dimensional Schrödinger equation as the exponential of a power series in h. Phase integral conditions are also obtained.

Cashwell, E. D. The asymptotic solutions of an ordinary differential equation in which the coefficient of the parameter is singular. Pacific J. Math. 1, 337-353 (1951); Math. Rev. 13, 461 (1952).

The differential equation

$$w''(s) - [\lambda^2 \sigma(s) + r(\lambda, s)]\, w(s) = 0$$

is studied, where $r(\lambda, s)$ has a pole of order one or two at s_0, λ is a large complex parameter, and $\sigma(s) = (s - s_0)^{-2}\, \psi(s)$, $\psi(s)$ being a single valued analytic function bounded away from zero. The author considers the normal form

$$u''(z) - \left[\frac{\rho^2 \phi^2(z) + A}{z^2} + X(\rho, z)\right] u(z) = 0,$$

where $X(z)$ is analytic and $\phi^2(z)$ is analytic and non-vanishing. Asymptotic forms of the solutions valid in certain regions of (λ, z) space are obtained in terms of the elementary functions by means of a related equation. In this connection see Langer, Trans. Amer. Math. Soc. 37, 397-416 (1935).

Cherry, T. M. Uniform asymptotic expansions. J. London Math. Soc. 24, 121-130 (1949): Math. Rev. 11, 34, (1950).

The author obtains uniform approximations to $J_\nu(x)$ to an arbitrarily high order in ν^{-1} for a range of the argument x which includes the transition point and is independent of ν. The method is one of comparison with the Airy equation by means of an integral equation.

Cherry, T. M. Uniform asymptotic formulae for functions with transition points. Trans. Amer. Math. Soc. **68**, 224-257 (1950); Math. Rev. **11**, 596, (1950).

Differential equations of the form

$$(*)\quad \frac{d^2y}{dz^2} + y\{-\nu^2 z + g(z, \nu^{-2})\} = 0,$$

where ν is a large complex parameter and $g(z, w)$ is analytic in both variables and is also regular at $z = 0$, $w = 0$ are studied. Approximations whose error is $O(\nu^{-n})$, uniformly for z in a closed region having the point $z = 0$ in its interior, are obtained by transforming the Airy equation into an equation approximately the same as $(*)$ and then comparing by means of an integral equation. Applications to Bessel and hypergeometric functions are given.

Cherry, T. M. Asymptotic expansions for the hypergeometric functions occurring in gas-flow theory. Proc. Roy. Soc. London Ser. A. **202**, 507-522 (1950); Math. Rev. **12**, 257 (1951).

The equation

$$(*)\quad \frac{d^2y}{dr^2} + \left(\frac{1}{r} \mp \frac{\beta}{1-r}\right)\frac{dy}{dr} + \nu^2\left[\frac{\beta^2}{2r(1-r)} - \frac{1}{4r^2}\right]y = 0,$$

where $0 \le r < 1$ and ν is large and complex is considered. The coefficient of y in $(*)$ has a simple zero at $r_s = 1/(1 + 2\beta)$. Non uniform asymptotic approximations to the solutions in terms of the elementary functions are obtained near r_s. Uniform asymptotic formulas involving Bessel functions are obtained by application of Cherry, Trans. Amer. Math. Soc. **68**, 224 (1950).

Goldstein, S. A note on certain approximate solutions of linear differential equations of the second order with an application to the Mathieu equation. Proc. London Math. Soc. (2) **28**, 81-90 (1928).

This paper extends the results of Jeffreys Proc. London Math. Soc. **23**, p. 428, to the case in which $X(x)$ has a zero of any order at $x = 0$. Formulas connecting the asymptotic forms on each side of $x = 0$ are obtained by using the asymptotic expansions of Bessel functions.

A correction of an error in this paper is given in Proc. London Math. Soc. 33, p. 246.

Goldstein, S. A note on certain approximate solutions of linear differential equations of the second order. Proc. London Math. Soc. (2) **33**, 246-252 (1932).

The author considers the equation

$$\frac{d^2y}{dx^2} - (h^2 X_0 + h X_1) y = 0,$$

where $X_0(x)$ has a double zero at the origin, and h is real and large. Connections between the asymptotic forms of the solutions on each side of $x = 0$ are established by means of the asymptotic representations of the parabolic cylinder functions $D_n(z)$.

Imai, Isao On a refinement of the W.K.B. method. Physical Rev. (2) **74**, 113 (1948); Math. Rev. **10**, 41 (1949).

The author studies the differential equation

$$\Phi'' + k^2 P(x) \Phi = 0,$$

where x is real, k is real and large, and $P(x)$ has a simple zero on the interval considered. An ingenious change of independent variable is used to obtain an improved approximate solution in the neighborhood of the zero of $P(x)$.

Jeffreys, Harold. On certain approximate solutions of linear differential equations of the second order. Proc. London Math. Soc. (2) **23**, 428-436 (1924-25).

The author considers the equation

$$\frac{d^2y}{dx^2} - h^2 X(x) y = 0,$$

where x is real, h is real and large, and $X(x)$ has a simple zero at x_0. Formulas connecting the asymptotic forms on each side of x_0 are obtained by means of the asymptotic expansions of $J_{1/3}(x)$ for large x.

Jeffreys, Harold. Asymptotic solutions of linear differential equations. Philos. Mag. (7) **33**, 451-456 (1942); Math. Rev. **4**, 43 (1943).

The author considers the following differential equation

$$\frac{d^2y}{dx^2} - (h^2 X_0 + h X_1 + X_2) y = 0,$$

where $X_0(x)$ has a simple zero at $x = 0$. Formulas connecting the asymptotic forms on each side of $x = 0$ are obtained with the aid of Airy functions.

Keller, H. B. and J. B. Keller. On systems of linear ordinary differential equations. New York University, Washington Square College Mathematics Research Group Research Report No. EM-33 (1951); Math. Rev. **13**, 346 (1952).

The system considered is

$$\frac{du(z)}{dz} = A(z)\, u(z),$$

where $A(z)$ is a square matrix, $u(z)$ a column vector, z ranges either over a real interval or a region in the complex plane. In domains in which $A(z)$ can be diagonalized, the Peano-Baker solution (matrizant) may be rearranged so as to form a rapidly convergent series. If $A(z) = kA_1(z)$ where k is a large parameter, this rearranged series may be converted into an asymptotic expansion (for large k) whose first term is the W.K.B. approximation. Turning points occur where the diagonal form of A breaks down. No uniform approximations are obtained for the neighborhood of such points. In the real case, connection formulas are derived by using the original Peano-Baker form in the neighborhood of a turning point and the modified formulas outside.

Kemble, Edwin C. A contribution to the theory of the W.K.B. method. Physical Rev. **48**, 549-561 (1935); **Fundamental Principles of Quantum Mechanics.** McGraw Hill (1937).

Zwaan's scheme for deriving the connection formulas at the transition point of

$$u'' + \lambda^2 \phi^2 u = 0$$

is put on a rigorous basis by considering the differential equations governing the variation in the coefficient during the fitting process.

Kramers, H. A. Wellenmechanik und halbzahlige Quantisierung. Z. Physik **39**, 828-840 (1926).

The differential equation

$$\phi'' + \frac{2\pi}{h} y \phi = 0,$$

where $y = y(x) = 2m\,[E_a - V(x)]$ is positive for $x_1 < x < x_2$ and negative elsewhere, is studied. Asymptotic forms of the solutions, valid for $x < x_1$,

$x_1 < x < x_2$, and $x > x_1$ are obtained in a heuristic manner by comparison with the Airy equation. The phase integral condition is also obtained.

Kumar Saha, Ajit. The transmission factors of potential barriers. Proc. Nat. Inst. Sci. India **10**, 373-385 (1944); Math. Rev. **9**, 436 (1948).

The differential equation studied is

$$(*)\quad \frac{d^2X}{dr^2}+\frac{2m}{\hbar^2}\left[E-V-\frac{l(l+1)}{r^2}\,\frac{\hbar^2}{2m}\right]X=0,$$

the coefficient of X being zero at two values of r and negative in between. Connection formulas which associate the asymptotic representations of (*) away from the transition points are obtained. The author develops the method of Langer for deriving these formulas and the results are compared with less satisfactory methods by Gamow, Sommerfeld and Bethe.

Langer, Rudolph E. On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order. Trans. Amer. Math. Soc. **33**, 23-64 (1931).

The differential equation studied is

$$(*)\quad \frac{d^2u}{dx^2}+p\frac{du}{dx}+(\rho^2\phi^2+q)u=0,$$

where ρ is a complex parameter, p, q, ϕ^2 are functions of x defined on an interval I which may be infinite, $\phi^2(x)$ has a zero at x_0 of order ν, and the function $(x-x_0)^{-\nu}\phi^2(x)$ is twice continuously differentiable and is real and positive apart from a constant complex factor. Asymptotic forms of the solutions are obtained by comparing (*) with a related equation whose solutions involve Bessel functions. As an application formulas are obtained for J_ρ, Y_ρ, H_ρ, with arguments $\rho\,\mathrm{sech}\,\alpha$ and $\rho\,\mathrm{sech}\,\beta$ for all real α and β and large positive ρ.

Langer, Rudolph E. On the asymptotic solutions of differential equations with an application to the Bessel functions of large complex order. Trans. Amer. Math. Soc. **34**, 447-480 (1932).

This paper is an extension of the results of Langer, Trans. Amer. Math. Soc. **33**, 23 (1931). The equation considered is

$$\frac{d^2y}{dz^2}+p(z)\frac{dy}{dz}+[\rho^2\phi^2(z)+q(z)]\,y(z)=0,$$

where z is a complex variable ranging over a simply connected region R_z of the complex plane, ρ is a complex parameter, and $\phi^2(z)=z^\nu\phi_1^2(z)$,

it being assumed that the origin lies in R_z, ν is a real non-negative number, $\phi_1^2(z)$ is an analytic function on R_z, and $|\phi_1(z)| \geq A > 0$ on R_z.

As an application the author obtains asymptotic formulas for Bessel functions of large complex order and argument.

Langer, Rudolph E. The asymptotic solutions of certain linear ordinary differential equations of the second order. Trans. Amer. Math. Soc. 36, 90-106 (1934).

The differential equation

$$\frac{d^2u}{ds^2} - [\lambda^2 q_0(s) + \lambda q_1(s) + q_2(s, \lambda)]u = 0$$

is studied in a region of the s plane in which $q_0(s)$ has a second order zero, the complex parameter λ being taken as large. Asymptotic forms of the solutions and connection formulas are obtained by comparison with the confluent hypergeometric function $M_{k,m}$.

Langer, Rudolph E. The solutions of the Mathieu equation with a complex variable and at least one parameter large. Trans. Amer. Math. Soc. 36, 637-695 (1934).

On the basis of his previous work in Trans. Amer. Math. Soc. (1931, 1932, 1934), the author studies the Mathieu equation

$$\frac{d^2u}{dz^2} + (\Lambda - \Omega \cos 2z)u = 0,$$

where z is complex and Λ and Ω are real parameters, at least one of which is large. The exterior of a circle in the (Ω, Λ) plane is divided into sectors and for each sector the asymptotic forms of the solutions in regions of the z plane are given. Connection formulas are also supplied.

Langer, Rudolph E. The asymptotic solutions of ordinary linear differential equations of the second order with special reference to the Stokes' phenomenon. Bull. Amer. Math. Soc. 40, 545-582 (1934).

This is a symposium lecture giving a general historical sketch of the problem of finding asymptotic solutions of the differential equation

$$\frac{d^2u}{dx^2} + [\lambda^2\phi^2(x) - X(x)]u = 0,$$

where $\phi^2(x)$ is first taken to be bounded away from zero and then is permitted to have a zero of order ν. References to the literature are given.

Langer, Rudolph E. On the asymptotic solutions of ordinary differential equations, with reference to the Stokes' phenomenon about a singular point. Trans. Amer. Math. Soc. **37**, 397-416 (1935).

The differential equation considered is

$$\frac{d^2w}{ds^2} + [\lambda\psi(s) + r(\lambda, s)]\,w = 0,$$

where λ is a large complex parameter, $r(\lambda, s)$ has a pole of order at most two at s_0, and $\psi(s) = (s - s_0)^\nu\,\psi_1(s)$, $\nu > -2$, with $\psi_1(s)$ a non-vanishing single-valued analytic function. The author considers the normal form

$$\frac{d^2u}{dz^2} + \left[\rho^2\phi^2(z) + \frac{\frac{1}{4} - A^2}{z^2} + X(\rho, z)\right] u = 0,$$

where X is analytic and $\psi^2(z) = z^{1/(\mu-2)}$ multiplied by an analytic function and $\mu = 1/[2(\nu + 2)]$. Asymptotic forms of the solution are obtained for regions of (ρ, z) space via Bessel functions.

Langer, Rudolph E. On the connection formulas and the solutions of the wave equations. Physical Rev. **51**, 669-676 (1937).

The author applies some of his previous results to the Schrödinger wave equation.

Langer, Rudolph E. The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point. Trans. Amer. Math. Soc. **67**, 461-490 (1949); Math. Rev. **11**, 438 (1950).

The differential equation considered is

$$(*)\qquad \frac{d^2u}{dx^2} + [\lambda^2 q_0(x) + \lambda q_1(x) + R(x, \lambda)]\,u = 0,$$

where $R(x, \lambda) = \sum_{\nu=0}^{\infty} r_\nu(x)/\lambda^\nu$, q_0, q_1, and r_ν each have derivatives of all orders, λ is large and complex, and $q_0(x)$ has a simple zero at the origin and is real on the interval (a, b). By means of Bessel's equation and successive transformations a sequence of approximating equations with known solutions is obtained. In this manner asymptotic solutions valid in $a \leq x \leq b$ are obtained, with an estimate of the error being obtained from an integral equation.

Langer, Rudolph E. On the wave equation with small quantum numbers. Physical Rev. (2) **75**, 1573-1578 (1949); Math. Rev. **10**, 710 (1949).

The author applies some of his previous work to the Schrödinger wave equation.

Langer, Rudolph E. Asymptotic solutions of a differential equation in the theory of microwave propagation. Comm. Pure Appl. Math. **3**, 427-438 (1950); Math. Rev. **12**, 828 (1951).

The author applies some of his previous work to a study of the equation

$$\frac{d^2u}{dh^2} + k^2[\Lambda + \gamma(h)]u = 0,$$

where Λ is a complex parameter.

Leavitt, William G. On systems of linear differential equations. Amer. J. Math. **73**, 690-696 (1951); Math. Rev. **13**, 346 (1952).

The author considers the equation

$$U' = (\lambda A + P)U$$

over the complex plane, where A and P are 2×2 matrices and U is a 2-vector. The equation is reduced to a canonical form in which

$$A = \begin{bmatrix} \mu & \psi \\ 0 & -\mu \end{bmatrix}$$

and ψ is either 0 or 1. Asymptotic solutions for large λ are given when μ has one single zero.

Lighthill, M. J. The hodograph transformation in trans-sonic flow. II. Auxiliary theorems on the hypergeometric functions $\psi_n(r)$. Proc. Roy. Soc. London Ser. A. **191**, 341-351 (1947); Math. Rev. **9**, 350 (1948).

The author applies a method of Langer, Trans. Amer. Math. Soc. (1931) to the following differential equation with transition point

$$\psi_n'' + A(r)\,\psi_n' + n^2 B(r)\,\psi_n = 0.$$

Meksyn, D. Asymptotic integrals of a fourth order differential equation containing a large parameter. Proc. London Math. Soc. (2) **49**, 436-457 (1947); Math. Rev. **9**, 436 (1948).

The following differential equation is considered:

$$\psi^{(iv)} + [-2a^2 + i\lambda(w - c)]\,\psi'' + [a^4 - i\lambda a^2(w - c) - i\lambda w'']\,\psi = 0,$$

where y is the independent variable, a^2 is a constant of order unity, λ is a large positive parameter, and $w - c$ is an even function of y defined on the range $-1 \leq y \leq 1$ and having a simple zero at y_0, $0 < y_0 < 1$. Asymptotic forms of the solutions on each side of y_0 are obtained together with connection formulas. The method is an extension of that of Jeffreys.

Schwid, Nathan. The asymptotic forms of the Hermite and Weber functions. Trans. Amer. Math. Soc. **37**, 339-362 (1935).

Asymptotic forms of the solutions of

$$\frac{d^2 w}{dz^2} + (2\kappa + 1 - z^2)\, w(z) = 0$$

which are valid for large complex parameter κ and all complex values of z are obtained. Asymptotic forms of the Hermite functions are also obtained. The method consists of an application of formulas of Langer.

Siefert, H. Zur asymptotischen Integration von Differentialgleichungen. Math. Z. **48**, 173-192 (1942); Math. Rev. **4**, 276 (1943).

The author considers the differential equation

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} - [\rho^2 q(x) + r(x)]\, y = 0,$$

where x is real, the functions, p, q, r, are real valued, and $q(x)$ has a simple zero at x_0. Given the asymptotic behavior of a solution on one side of x_0, the author determines its behavior on the other side by analytic continuation. No results in the transition region are given.

Sips, Robert. Représentation asymptotique des fonctions de Mathieu et des fonctions d'onde sphéroidales. Trans. Amer. Math. Soc. **66**, 93-134 (1949); Math. Rev. **11**, 435 (1950).

The differential equations

$$\left(1 - \frac{a^2}{2\lambda c}\right) y'' - \frac{a}{2\lambda c}\, y' + \left(A - \frac{a^2}{4}\right) y = 0 \qquad \text{(Mathieu)}$$

$$\left(1 - \frac{a^2}{2\lambda c}\right) y'' - \frac{m+1}{\lambda c}\, a y' + \left(K - \frac{a^2}{4}\right) y = 0 \qquad \text{(Spheroidal)}$$

are compared with $\eta'' + (n + \tfrac{1}{2} - \tfrac{1}{4} a^2)\,\eta = 0$, and the equation

$$\left(1 - \frac{a^2}{2\lambda c}\right) y'' + \left(\frac{1}{a} - \frac{a}{2\lambda c}\right) y' + \left(\frac{A}{2\lambda c} - \frac{a^2}{4} - \frac{m^2}{a^2}\right) y = 0$$

(Spheroidal)

is compared with

$$\eta'' + \frac{1}{a}\,\eta' + \left(m + 2\rho + 1 - \frac{a^2}{4} - \frac{m^2}{a^2}\right)\eta = 0.$$

Strutt, M. J. O. Characteristic curves of Hill problems. II. The asymptotic form of the curves. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 212-222 (1943); Math. Rev. 7, 160 (1946).

The differential equation considered is

$$\frac{d^2w}{dz^2} + [p(z) + \lambda + \gamma\phi(z)]\, w = 0$$

where z is real, $(|\lambda| + |\gamma|)\, \zeta^2$ is large, $p(z)$ and $q(z)$ are periodic with period ζ, and $p(z) + \lambda + \gamma\phi(z)$ has two simple zeros in $z_0 < z < z_0 + \zeta$. Asymptotic formulas for the solutions are obtained by an application of Langer, Trans. Amer. Math. Soc. (1934) and Bull. Amer. Math. Soc. (1934).

Taylor, W. C. A complete set of asymptotic formulas for the Whittaker function and the Laguerre polynomials. J. Math. Physics 18, 34-49 (1939).

The author applies the methods of Langer, Trans. Amer. Math. Soc. (1931, 1932, 1935), to the equation

$$\frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) u = 0,$$

where k, m, z are complex, m is bounded, $k \to \infty$, and z is unrestricted.

Timman, R. Asymptotic formulae for special solutions of the hodograph equation in compressible flow. Nationaal Luchtvaartlaboratorium, Amsterdam, Report F. 46, i + 26 pp. (1949); Math. Rev. 10, 711 (1949).

The differential equation considered is

$$(*)\quad \frac{d^2 W}{d\tau^2} - \left[\frac{n^2 - 1}{4}\,\frac{1-\tau/\tau_s}{\tau^2(1-\tau)} + \frac{(1-\tau_s)(1+3\tau_s)}{16(1-\tau)^2\,\tau_s^2}\right] W = 0,$$

where n is a parameter and τ_s is a constant. The author adapts the method of Langer, Trans. Amer. Math. Soc. 34, 441-480 (1932), and transforms the problem of finding the solutions of (*) into a problem of solving a certain integral equation. He then obtains a convergent Neumann series for the solutions, which for large values of n leads to simple asymptotic formulas for the solution. The formulas for $\tau < \tau_s$ are different from those for $\tau > \tau_s$.

Tricomi, Francesco. Un nuovo metodo di studio delle equazioni differenziali lineari. Univ. e Politecnico Torino Rend. Sem. Mat. 8, 7-19 (1949); Math. Rev. 11, 437 (1950).

This is an expository article devoted to a method of Fubini for solving

equations of the form

$$y'' + p_1(x)y' + p_2(x)y = A(x)y'' + B(x)y' + C(x)y,$$

where two independent solutions of the left hand side are known. The method is one of reduction to two integral equations of the Volterra type which can be solved by successive approximations. If $A(x) = O(\nu^{-r})$, $B(x) = O(\nu^{-r})$, $C(x) = O(\nu^{-r})$, $r > 0$, then this method gives asymptotic forms for the Laguerre polynomials $L_m(x)$, $m \to \infty$. The zeros of $L_m(x)$ are also considered.

Tricomi Francesco. Sul comportamento asintotico dell' n-esimo polinomio di Laguerre nell' intorno dell' ascissa 4n. Comment. Math. Helv. **22**, 150-167 (1949); Math. Rev. **10**, 703 (1949).

The author applies the method of Fubini to the confluent hypergeometric equation

$$xy'' + (c - x)y' - ay = 0$$

to obtain the asymptotic behavior of the Laguerre polynomials $L_m^{(a)}(x)$ in the vicinity of $x = 4m + 2(a + 1)$ as $m \to \infty$.

Tricomi, Francesco. Sulle funzioni di Bessel di ordine e argomento pressochè uguali. Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **83**, 3-20 (1949); Math. Rev. **11**, 594 (1950).

The following asymptotic representation is obtained

$$\pi J_\nu[\nu + (\tfrac{1}{6}\nu t)^{1/3}] = (\tfrac{1}{6}\nu)^{-1/3} A_1(t) + (10\nu)^{-1}[3t^2 A_1'(t) + 2t A_1(t)] + O(\nu^{-5/3}),$$

where ν is real and large, t is real, $A_1(t)$ is Airy's function,

$$\tfrac{1}{3}\pi(\tfrac{1}{3}t)^{1/2}\{J_{-1/3}[2(\tfrac{1}{3}t)^{3/2}] + J_{1/3}[2(\tfrac{1}{3}t)^{3/2}]\}.$$

A similar formula is obtained for Bessel's function of the second kind and a discussion of the zeros of Bessel functions of the first and second kind is given. The method used is that of Fubini; see Tricomi, Rend. Sem. Mat. 8, 7-19 (1948).

Wasow, Wolfgang. On the asymptotic solution of the differential equation for small disturbances in a laminar flow. Proc. Nat. Acad. Sci. U.S.A. 33, 232-234 (1947); Math. Rev. 9, 144 (1948).

The differential equation

$$y^{(iv)} + a_3 y^{(iii)} + \sum_{k=0}^{2} (\lambda^2 b_k + a_k) y^{(k)} = 0$$

is considered over a domain in the complex plane in which the coefficients are analytic and in which b_2 has a simple zero. The parameter λ is taken to be of constant argument. The author sets $Q(x) = \int [-b_2(x)]^{1/2} dx$ and divides a neighborhood of the zero of b_2 into three sectors by means of the curves $\Re[\lambda Q(x)] = 0$. He then states without proof four theorems on the asymptotic representation of solutions in these sectors.

Wasow, Wolfgang. A study of the solutions of the differential equation $y^{(4)} + \lambda^2(xy'' + y) = 0$ for large values of λ. Ann. of Math. (2) 52, 350-361 (1950); Math. Rev. 12, 261 (1951).

The author considers the equation

$$y^{(iv)} + \lambda^2 (xy'' + y) = 0,$$

where x is complex and λ is large and real. Asymptotic forms of the solutions are found from integral representations by the method of steepest descent.

Wasow, Wolfgang. Asymptotic solution of the differential equation of hydrodynamic stability in a domain containing a transition point. National Bur. Standards, Report 1620 (Preprint) 63 pp. (1952).

The differential equation considered is

$$u^{(iv)} + \sum_{i=1}^{4} a_i(x) u^{(4-i)} + \lambda^2 \sum_{i=0}^{2} b_i(x) u^{(2-i)} = 0,$$

where x is complex, $b_0(x)$ has one simple zero at $x = 0$, $b_1(x) = 0$, $b_2(0) \neq 0$. The author finds functions asymptotic to a fundamental system of solutions in a fixed neighborhood of $x = 0$ as $\lambda \to \infty$. The method resembles that of Langer. Use is made of the known asymptotic solutions of $y^{(iv)} + \lambda^2(xy'' + y) = 0$. The transformation between the two spaces of solutions is not linear.

Wasow, Wolfgang. On the differential equation for the stability of plane Couette flow. National Bur. Standards (working paper 1952).

The differential equation studied is

$$u^{(iv)} - 2a^2 u'' + a^4 u + \lambda^2 x(u'' - a^2 u) = 0.$$

From the contour integral representations of a fundamental set of solutions $[u_i(x, \lambda)]$, the following results are obtained

$$u_i = \begin{cases} g_i(x, \lambda)\,[1 + O(\xi^{-1})] & x \text{ in } S, \quad |\xi| \geq \xi_0 > 0 \\ h_i(x, \lambda)\,[1 + O(\lambda^{-4/3})] & x \text{ in } S \quad |\xi| \leq \xi_0 \end{cases}$$

where $\xi = \lambda x$, S is a bounded neighborhood of $x = 0$, and the functions g_i and h_i are well known. A similar discussion for the equation

$$u^{(iv)} + \lambda^2 x(u'' + u) = 0$$

is given.

Wentzel, Gregor. Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik. Z. Physik **38**, 518-529 (1926).

A formal expansion in powers of $h/2\pi i$ of the solution of the associated Riccati equation is used to obtain the eigenvalues of the Schrödinger equation.

Zwaan, A. Intensitäten im Ca. Funkenspectrum. Thesis - Utrecht (1929).

The differential equation considered is

$$u'' + \lambda^2 \phi^2(x)\, u = 0,$$

where ϕ^2 is assumed to have a simple zero at the origin, is positive on the positive real axis, and admits of an analytic approximation within a suitable region of the complex plane containing the origin in its interior. For large negative values of x the following solution is obtained

$$(*) \quad u(x) = \frac{c_1 e^{i\xi}}{\phi^{1/2}} + \frac{c_2 e^{-i\xi}}{\phi^{1/2}} \qquad \xi = \lambda \int_0^x \phi\, du.$$

Let Γ denote a semi-circular arc with center at the origin and lying in the upper half plane and such that at each point of Γ a representation of the form $(*)$ for the solution holds. By following the variations in $(*)$ as the arc Γ is traversed from the negative real axis to the positive real axis asymptotic representations of the solutions valid on the positive side of the origin are obtained.

Langer, Bull. Amer. Math. Soc. **40**, 562 (1934) and Kemble, Fundamental Principles of Quantum Mechanics, McGraw Hill (1937), both discuss this method.